Recovering the Probability Weights of Tail Events with Volatility Risk from Option Prices

Fousseni Chabi-Yo and Zhaogang Song
Fisher College of Business and Federal Reserve Board

Second ITAM Finance Conference 2013
Mexico City, June 7-8th 2013

June 8, 2013

What is the Big Picture?

1. In a simple model, we show that investors over-weight the tails of the joint distribution of the S&P500 index and VIX.

2. We use options on S&P500 index and options on VIX, and estimate by how much investors over-weight small probabilities of bad outcomes.

3. Our estimates show that the joint probability weighting function of both return on SP500 and VIX is stable over time, while that of the return is time-varying depending on VIX.

4. We show that investor time-varying tail weighting attitudes toward the likelihood of disaster risks in equity and option markets can help predict the market return, and also explain tail risk premium.

Do we care about over-weighting potential outcomes?

- "If an individual is aware of a potential tail event, he will overweight this potential outcome in his decision- making relative to the weight that the outcome would receive in the expected utility framework "(Barberis (2013))

- Individual will use a transformation G of the true physical probability P

$$\underbrace{G[P]}_{\text{Probability Weighting Function}} = \underbrace{G[P] - P}_{\text{Over-weight Probability}} + \underbrace{P}_{\text{True Probability}}$$

- The probability weighting function $G[.] : [0, 1] \rightarrow [0, 1]$ satisfies $G[0] = 0$ and $G[1] = 1$. G is differentiable, continuous, and non-decreasing. $G'[P] = Z[P]$ is the density weighting function.

Non-Expected Utility Framework

- Hypothesis: in equity and option markets, investors use non-expected utility models to maximize their utility.
- Few examples of non-expected utility models:
  - Rank-Dependent Expected Utility model in Quiggin (1993)
  - The cumulative prospect theory: Tversky and Kahneman (1992), Barberis and Huang (2008).
- Main idea in equity and option markets: investors overweight small probabilities of bad outcomes.

Findings

1. The probability weighting function of the S&P500 return is inverse S-shaped and is time-varying: Investors **over-weight the tails** of the return's distribution.

2. The joint probability weighting function of both S&P500 return and VIX is **stable over time**, while that of the return is time-varying depending on VIX.

3. Ignoring the volatility risk may lead to a severe bias so that one may conclude that investors under-weight tails of the returns distribution while they in fact overweight.

4. Tail event measures based on the probability weighting function vary over time, and have **strong return predictability and explanatory power for tail risks up to a one-year horizon.**

How different is Our Contribution Compared to the Literature?

1. Dierkes (2009) and Polkovnichenko and Zhao (2012) estimate **probability weighting functions of the S&P500 returns assuming that the Stochastic Discount Factor (SDF) depends only on the S&P500 return: The Stochastic Discount Factor is not affected by the volatility risk factor**. .
2. A number of paper have shown that the SDF depends on volatility risk: **Chabi-Yo (2012), Bakshi, Madan and Panayotov (2010), Christoffersen, Heston, and Jacob (2010) and more recently Song and Xiu (2012)**.
3. Both Dierkes (2009) and Polkovnichenko and Zhao (2012) do not investigate whether over-weighting probability predicts the market return and tail risk premium.

Why Previous Papers Miss the Volatility in the SDF Specification?

- Consider a one-period model in which the rank dependent expected utility investor maximizes her utility conditional on the available information set $\mathcal{F}_t$ at time t

$$U = \mathbb{E}\left(u\left[W_T\right] Z\left[P\left[.|\mathcal{F}_t\right]\right] | \mathcal{F}_t\right)$$

where

$$Z\left[P\left[.|\mathcal{F}_t\right]\right] = G^{'}\left[P\left[.|\mathcal{F}_t\right]\right]$$

- Up to a constant, the pricing kernel consistent with the Euler Equation is

$$m_T = \frac{1}{R_t^f} u^{'}\left[W_T\right] Z\left[P\left[.|\mathcal{F}_t\right]\right] = \frac{1}{R_t^f} \frac{p^*\left[R_T|\mathcal{F}_t\right]}{p\left[R_T|\mathcal{F}_t\right]}.$$

The probability weighting function is

$$G\left[P\left[R_T|\mathcal{F}_t\right]\right] = \int_0^{R_T} Z\left[P\left[x|\mathcal{F}_t\right]\right] p\left[x|\mathcal{F}_t\right] dx.$$

A simple model

The Rank Dependent Expected Utility investor maximizes her expected utility over **two periods**, $[t, T+1] = [t, T] \cup [T, T+1]$, by solving

$$\max_{\{\phi_{kt}, \phi_{k,T}\}} \mathbf{U} \tag{1}$$

subject to

$$\mathbf{U} = \mathbb{E}\left(u\left(W_{t,T+1}\right) Z\left[P\left[.|\mathcal{F}_t\right]\right] | \mathcal{F}_t\right),$$

- $Z\left[P\left[.|\mathcal{F}_t\right]\right]$ is the density weighting function defined over the time period $[t, T+1]$.
- The RDEU investor's terminal wealth reads as

$$W_{t,T+1} = W_T W_{T+1},$$

- When $Z\left[P\left[.|\mathcal{F}_t\right]\right] = 1$, our framework reduces to EU framework.

A simple model

- Up to a constant, the SDF takes the form

$$m_T = \frac{1}{R_t^f} V_T Z\left[P\left[.|\mathcal{F}_t\right]\right]$$

where

$$V_T = \mathcal{A}_0 + \mathcal{A}_1 \left(\frac{S_T}{S_t} - \overline{a}\right) + \mathcal{A}_2 \left(\sigma_T^2 - \mathbb{E}\left[\sigma_T^2\right]\right)$$

- **Under no-arbitrage restrictions:** we have

$$m_T = \frac{1}{R_t^f} \frac{p^*\left[S_T, \sigma_T | \mathcal{F}_t\right]}{p\left[S_T, \sigma_T | \mathcal{F}_t\right]}.$$

Recovering the Probability Weighting Function

Set the RDEU SDF to the model-free SDF

$$V_T Z\left[P\left[R_T, \sigma_T | \sigma_t\right]\right] = \frac{p^*\left[S_T, \sigma_T | \mathcal{F}_t\right]}{p\left[S_T, \sigma_T | \mathcal{F}_t\right]},$$

1. Probability weighting function of both the return and volatility

$$G[P\left[R_T, \sigma_T^2 | \sigma_t\right]] = \int_0^{R_T} \int_0^{\sigma_T^2} Z\left[P\left[x, y | \sigma_t\right]\right] p\left[x, y | \sigma_t\right] dxdy$$

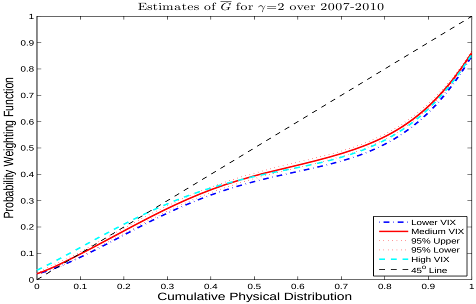
2. Marginal probability weighting function of the return

$$G[P\left[R_T | \sigma_t\right]] = \int_0^{\sigma_{max,T}^2} Z\left[P\left[x, y | \sigma_t\right]\right] p\left[x, y | \sigma_t\right] dy$$

Non-Parametric Estimates of the Probability Weighting Function (PWF)

1. **Non-Parametric Estimates of PWF**

Figure : Non-Parametric PWF $\overline{G}[\overline{P}[R_T, \sigma_T|\sigma_t]]$



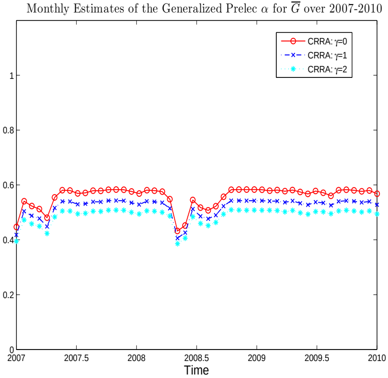
2. Parametric PWF (**Generalized Prelec Function**)

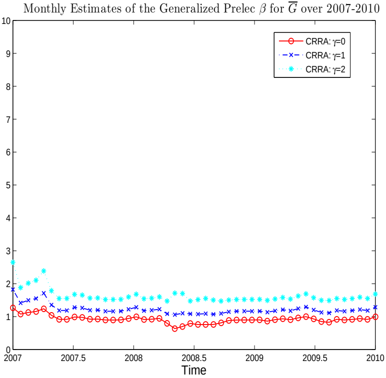
$$G^{GPL}[P] = \exp\left(-\left(-\log\left(P^{\beta}\right)\right)^{\alpha}\right).$$

Fitting the Generalized Prelec Function to Non-Parametric PWF

What can we learn?

- The time-series dynamic of α and β shows little variation in the PWF
- Investor attitudes toward extreme events do not change over time!

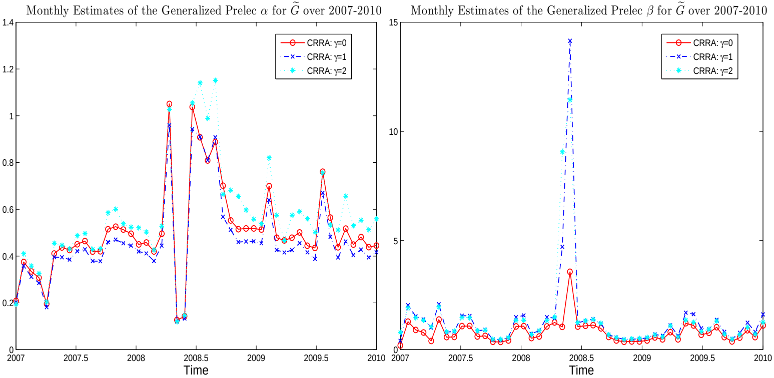




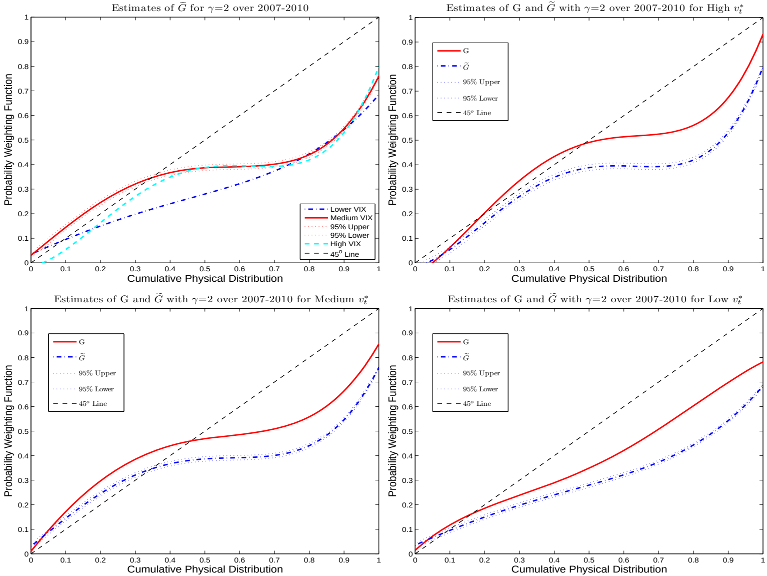
PWF of Return Only

What can we learn?

- The time-series dynamic of α and β shows strong variation in PWF
- Investor attitudes toward extreme events in the tail of the index return change over time!



Ignoring Volatility Risk leads to a Bias in the PWF of Return Only



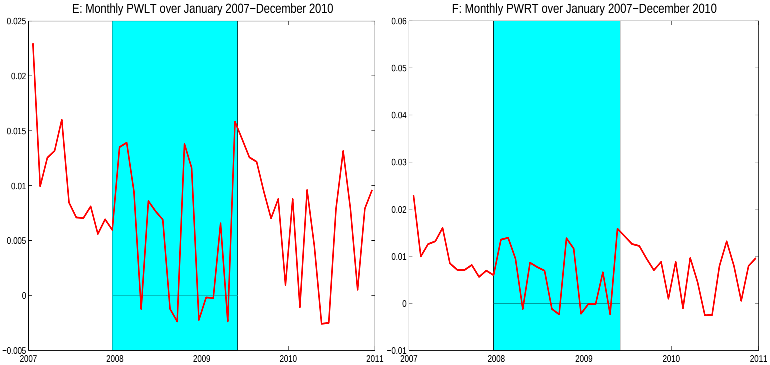
Non-Parametric Weighting Measures of Tail Events

- We focus on the PWF of return only
- Two measures of tail event risks beyond existing measures of tail risks

$$PWLT = \int_0^{P_0} \left(\widetilde{G}\left[P\right] - P \right) dP \text{ and } PWRT = \int_{1-P_0}^{1} \left(P - \widetilde{G}\left[P\right] \right) dP$$

- Our measures are distinct from existing measures. Existing measures focus on the likelihood of rare-disaster risks. We focus on the amount of over-weighting the probability of a disaster.
- Our measures are not substitutes, but complement existing measures.
- In our updated version, we also focussed on the PWF of VIX.

Time-Varying PWF Index-Based Measures and the Business Cycle



Predicting the Market Return with 56-day PWLT and PWRT

- **The PWLT and PWRT measures strongly predict the market return at short horizons**

Horizons	1-month	3-month	6-month	12-month
Const	-0.47	-0.09	-0.11	-0.04
	(-1.70)	(-0.54)	(-0.72)	(-0.23)
PWLT	**39.39**	**28.23**	**16.28**	-2.97
	(2.22)	**(2.77)**	**(1.89)**	(-0.55)
PWRT	-1.41	**-14.43**	**-7.96**	-3.06
	(-0.22)	**(-3.65)**	**(-2.74)**	(-1.37)
VRP	0.84	0.45	0.29	0.09
	(2.37)	(2.38)	(2.19)	(0.93)

Predicting the Market Return with 56-day PWLT and PWRT

- Results are robust to a battery of predictor variables (P/E ratio, Variance Risk Premium, Term Spread, Default Spread, etc)

Do Probability Weighting Index-Based Measures Capture Tail Risk Premium?

- PWLT and PWRT explain the tail risk measure of Du and Kapadia (2011)

$$\text{JTIX}_t = \underset{(7.85)}{0.04} - \underset{\textbf{(-2.74)}}{\mathbf{1.17}}\,\text{PWLT}_t - \underset{\textbf{(-2.15)}}{\mathbf{0.60}}\,\text{PWRT}_t \qquad \text{R}^2 = 27.81\%$$

- PWLT and PWRT explain the variance risk premium that incorporates tail risk premium of Du and Kapadia (2011)

$$\text{VRP-DK}_t = \underset{(3.26)}{0.09} - \underset{\textbf{(-2.26)}}{\mathbf{1.35}}\,\text{PWLT}_t - \underset{\textbf{(-4.68)}}{\mathbf{1.75}}\,\text{PWRT}_t \qquad \text{R}^2 = 24.76\%$$

Conclusion

1. We show that investors over-weight the likelihood of extreme events in the joint distribution of the S&P500 index and VIX.
2. We use both options on S&P500 index and VIX, and estimate by how much investors over-weight the likelihood of rare events in equity and option markets.
3. We find that the joint probability weighting function of both return on SP500 and VIX is stable over time, while that of the return is time-varying depending on VIX.
4. We show that investor time-varying attitudes toward the likelihood of disaster risks in equity and option markets can help predict the market return at short horizons, and also explain tail risk premium.